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Homotopical properties of upper semifinite hyperspaces of compacta

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Abstract

In this paper we study homotopical properties of a special neighborhood system, which is denoted by $\{U_\varepsilon\}_{\epsilon>0}$, for the canonical embedding of a compact metric space in its upper semifinite hyperspace to get results in the shape theory for compacta. We also point out that there are spaces with the shape of finite discrete spaces and having not the homotopy type of any T_1-space

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1. Introduction

Let us denote by 2^X the hyperspace of non-empty closed sets with the upper semifinite topology.

In this paper we want to show how a non-T_1 topology in hyperspaces, the upper semifinite topology, can be used to study and reformulate some geometrical aspects of the topology of compact metric spaces. Our point of view is to consider the canonical copy of a compactum X inside the upper semifinite hyperspace (denoted by 2^X). This hyperspace is highly non-Hausdorff, in fact 2^X is a compactification of X, being X also a compact space. In a previous paper [2] the authors describe some properties of 2^X when X is a normal Hausdorff space.

All along this paper when we refer to the upper semifinite hyperspace 2^X it is meant that X is a compact metric space.

We begin Section 2 considering a compact metric space (X, d) and establishing that the family $\mathcal{U} = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of the canonical copy of X inside 2^X, where $U_\varepsilon = \{C \in 2^X \mid \operatorname{diam}(C) < \varepsilon\}$ and diam represents the diameter function for the metric d. This was first proved in [1] for the hyperspace 2^X_H with the Vietoris topology, induced by the Hausdorff metric related to d. Later on we prove that 2^X is an absolute extensor for the class of compact metric spaces (in the sense of [6, p. 35]). We finish this section giving a *homotopy extension property for* 2^X.

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In Section 3 we study some shape and homotopy properties of the open sets U_ε, with $\varepsilon > 0$. In fact we prove that the least upper bound of the cardinal of the image of any continuous function from U_ε to any T_1-space is an integer which determines the shape of U_ε. One of the main results obtained in this section is that the upper semifinite hyperspace can be used to detect the shape morphism induced by a map. As a consequence we point out one of the main differences between shape and homotopy for topological spaces: *While each U_ε has the shape of some discrete finite space, some of them have not the homotopy type of any T_1-space*. On the other hand K. Morita [9] proved that each topological space has the same shape of some Tychonov space.

In the last section we use the hyperspace, with the upper semifinite topology to reinterpret Sanjurjo's description of shape of compact metric space [10] and, at the same time, we obtain that upper semifinite hyperspaces are as good ambient spaces as the Hilbert cube (used by K. Borsuk in [5]) to define shape theory.

We have to say that in [3, pp. 2784–2785] there are some results which are weakly related to a minor part of this paper.

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2. Extension properties in upper semifinite hyperspaces of compacta

Suppose that (X, d) is a compact metric space and consider the hyperspace 2^X of non-empty closed sets with the upper semifinite topology. Given an open subset $U \subset X$ we define $B_U = \{C \in 2^X \mid C \subset U\}$. Then the family $\mathcal{B} = \{B_U\}_{U \in T}$ is a base for the upper semifinite topology on 2^X. Also we have that if $C \in 2^X$ then $\overline{\{C\}} = \{D \in 2^X \mid C \subset D\}$. We can identify, topologically, (X, d) with the subspace $\phi(X) \subset 2^X$, where

$$\phi : X \longrightarrow 2^X, \qquad x \longmapsto \{x\}$$

is the so-called canonical embedding.

The following result will be very useful and widely used along the paper.

Proposition 1. *Let (X, d) be a compact metric space. Then the family $\mathcal{U} = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of X inside 2^X, where $U_\varepsilon = \{C \in 2^X \mid \operatorname{diam}(C) < \varepsilon\}$. Consequently $\{U_{1/n}\}_{n\in\mathbb{N}}$ is countable base.*

Proof. Obviously $X \subset U_\varepsilon$, for all $\varepsilon > 0$. Firstly we are going to prove that U_ε is an open neighborhood of X for each $\varepsilon > 0$.

Let $C \in U_\varepsilon$ with $\operatorname{diam}(C) = r < \varepsilon$. Consider $B(C, \frac{\varepsilon-r}{2}) = \{x \in X\colon d(x, C) < \frac{\varepsilon-r}{2}\}$ and take the open neighborhood $B_{B(C,\frac{\varepsilon-r}{2})}$ of C in 2^X. Let $D \in B_{B(C,\frac{\varepsilon-r}{2})}$ and $p, p' \in D$. Then we have

$$d(p, p') \leqslant d(p, C) + d(p', C) + \operatorname{diam}(C) < \varepsilon,$$

so $D \in U_\varepsilon$ and hence $B_{B(C,\frac{\varepsilon-r}{2})} \subset U_\varepsilon$.

Take now an open set $U \subset 2^X$ with X $(\equiv \phi(X)) \subset U$. For each $x \in X$ choose $\varepsilon_x > 0$ such that $V_x = B_{B(x,\varepsilon_x)} \subset U$. Choose a Lebesgue number $\beta > 0$ for the open cover $\{V_x\colon x \in X\}$. Then for each $x \in X$, $B_{(B(x,\beta))} \subset U$.

Consider $U_\beta = \{C \in 2^X \mid \operatorname{diam}(C) < \beta\}$, fix $C \in U_\beta$ and $c \in C$. Then $C \in B_{B(c,\beta)} \subset B_{B(x,\varepsilon_x)}$, for some $x \in X$. Consequently, $U_\beta \subset U$. $\square$

Proposition 2. *Let X be a compact metric space. Then 2^X is an absolute extensor for the class of compact metric space.*

Proof. Let (Y, d') be a compact metric space and $A \subset Y$ a closed set of Y. For each $y \in Y$ we define the set $A_y = \{a \in A \mid d'(a, y) = d'(y, A)\}$. Let $\{a_n\}_{n\in N} \subset A_y$ be a convergent sequence with $\lim_{n\to\infty} a_n = a_0 \in A$. Since $d'(a_n, y) = d'(A, y)$ for all $n \in \mathbb{N}$, it follows that $d'(y, A) = \lim_{n\to\infty} d'(y, a_n) = d'(y, a_0)$. Consequently $a_0 \in A_y$, and so A_y is a closed subset of A.

Consider the upper semifinite hyperspace 2^A. We define the function $g : Y \to 2^A$ by $g(y) = A_y$. Suppose g is not continuous at $y_0 \in Y$. Consider $B(A_{y_0}, \varepsilon) = \{a \in A \mid d'(a, A_{y_0}) < \varepsilon\}$ an open neighborhood of A_{y_0} in A, so $\{B_{B(A_{y_0},\varepsilon)}\}_{\varepsilon>0}$ is a base for the point A_{y_0} in 2^A and take $\mathcal{B} = \{B(y_0, \varepsilon_n) \mid \lim_{n\to\infty} \varepsilon_n = 0\}$ a base for the point y_0 in Y.

Then there exists $\varepsilon_0 > 0$ such that $g(B(y_0, \varepsilon_n)) \not\subset B_{B(A_{y_0}, \varepsilon_0)}$ for any $\varepsilon_n > 0$. For each $n \in \mathbb{N}$ we choose $y_n \in B(y_0, \varepsilon_n)$ such that $g(y_n) \not\subset B_{B(A_{y_0}, \epsilon)}$. So it follows that for each $n \in \mathbb{N}$ there exists $b_n \in g(y_n)$ such that $d'(b_n, A_{y_0}) \geqslant \epsilon_0$ and $d'(y_n, b_n) = d'(y_n, A)$. The sequence $\{b_n\}_{n\in N}$ has a convergent subsequence, we denote it again by $\{b_n\}_{n\in\mathbb{N}}$. Suppose that $\lim_{n\to\infty} b_n = b$. Since $\lim_{n\to\infty} d'(b_n, A_{y_0}) = d'(b, A_{y_0}) \geqslant \epsilon_0$ and $d'(y_0, A) \leqslant d'(y_0, b_n) \leqslant d'(y_0, y_n) + d'(y_n, b_n)$ then we have $d(y_0, A) \leqslant d(y_0, b) \leqslant d(y_0, A)$. Therefore $b \in A_{y_0}$, which is impossible.

Let $f : A \to 2^X$, be a continuous function and let $C \in 2^A$. We shall show that $\bigcup_{y\in C} f(y)$ is a closed set in X. Let $\{x_n\}_{n\in N} \subset \bigcup_{y\in C} f(y)$ be a convergent sequence in X. So for each $n \in \mathbb{N}$, there exists $y_n \in C$ such that $x_n \in f(y_n)$. Then the sequence $\{y_n\}_{n\in N} \subset C$ contains a subsequence, we denote it again by $\{y_n\}_{n\in N} \subset C$, which converges to $y_0 \in C$. Take now $B_{B(y_0,\varepsilon)} \subset 2^X$. By continuity of f it follows that there exists $n_0 \in \mathbb{N}$ such that $f(y_n) \in B_{B(y_0,\varepsilon)}$ for every $n \geqslant n_0$. Consequently $d(x_0, f(x_0)) < \epsilon$ for every $\varepsilon > 0$ and then $x_0 \in f(y_0)$. So $\bigcup_{y\in C} f(y) \in 2^X$.

Define the function $h : 2^A \to 2^X$ by $h(C) = \bigcup_{y\in C} f(y)$. Let B_V be an open neighborhood of $h(C)$ in 2^X. Evidently $B_{f^{-1}(B_V)}$ is open in 2^A. Then we have $C \subset f^{-1}(h(C)) \subset f^{-1}(B_V)$ and so $C \in B_{f^{-1}(B_V)}$. Take now $D \in B_{f^{-1}(B_V)}$. Then for each $d \in D$ it follows that $f(d) \in B_V$ and so $h(D) \in B_V$. Therefore $h(B_{f^{-1}(B_V)}) \subset B_V$ and we conclude that h is a continuous function.

Finally we define the map $f^* : Y \to 2^X$ by $f^* = h \circ g$, which is a continuous extension of f. $\square$

The next theorem describes a kind of homotopy extension property and is, in some sense, similar to the Borsuk's Homotopy Extension Theorem for ANRs (see [4] or [6, Chapter IV]). First of all from the last proof we can extract the following:

Lemma 3. *Let Z and T be compact metric spaces and suppose that $h : Z \to 2^T$ is a continuous function. For every $C \in 2^Z$, $\bigcup_{c\in C} f(c) \in 2^T$ and $h^* : 2^Z \to 2^T$ given by $h^*(C) = \bigcup_{c\in C} f(c)$, is continuous.*

In the next result we use the construction U_ε for two metric spaces X and Y, so we will denote them by $U_\varepsilon(X)$ and $U_\varepsilon(Y)$, respectively.

Theorem 4. *Let X, Y be two compact metric spaces. Suppose that $H : X \times I \to 2^Y$ and h: $2^X \to 2^Y$ are continuous functions such that $H(x, 0) = h_{|X}(\{x\})$. Then there exists a continuous function $\widetilde{H} : 2^X \times I \to 2^Y$ with the following properties*:

(i) $\widetilde{H}(C, 0) = h(C)$ *for all* $C \in 2^X$.
(ii) $\widetilde{H}_{|X\times I} = H$.
(iii) $\widetilde{H}$ *is continuous.*
(iv) *If $H(x, t) \in U_\varepsilon(Y)$ for all $(x, t) \in X \times I$, then we can choose $\delta > 0$ such that $\widetilde{H}(U_\delta(X) \times I) \subset U_\varepsilon(Y)$.*

Proof. First of all there is a natural topological embedding of $2^X \times I$ into $2^{X\times I}$ given by $(C, t) \mapsto C \times \{t\}$ for any $C \in 2^X$ and $t \in I$. Using the previous lemma we have the continuous extension $H^* : 2^{X\times I} \to 2^Y$ of H.

We define the function $\widetilde{H} : 2^X \times I \to 2^Y$ by

$$\widetilde{H}(C, t) = \begin{cases} h(C) & \text{if } t = 0, \\ H^*(C \times \{t\}) & \text{if } t > 0. \end{cases}$$

By the hypotheses we have that $\widetilde{H}$ is well defined and satisfies the properties (i)–(ii). Let us show that $\widetilde{H}$ also satisfies (iii)–(iv).

(iii) We need only to prove continuity at $(C, 0)$. Let B_V be a basic neighborhood of $\widetilde{H}(C, 0) = h(C)$ in 2^Y. Since h is continuous, there exists a basic neighborhood B_U of C in 2^X such that $h(B_U) \subset B_V$. In particular $h(\{c\}) \in B_U$ for each $c \in C$. Moreover H is also continuous and $H(c, 0) = h(\{c\})$, then we have that for each $c \in C$ there exists an open neighborhood $U_c \times [0, \varepsilon_c)$ of $(c, 0)$ in $X \times I$ such that $H(U_c \times [0, \varepsilon_c)) \subset B_V$. We can also suppose that $U_c \subset U$ for all $c \in C$. Since $\{U_c \mid c \in C\}$ is an open cover of C, then we can suppose that $\{U_{c_i} \mid i \in \{1, \dots, k\}\}$ is a finite subcover. Let $\varepsilon = \min\{\varepsilon_{c_i} \mid i \in \{1, \dots, k\}\}$ and $\widetilde{U} = (\bigcup_{i=1}^k U_{c_i})$. Take the open subset $B_{\widetilde{U}} \times [0, \varepsilon)$ of $2^X \times I$. Using the definition of H^* we easily see that $\widetilde{H}(B_{\widetilde{U}} \times [0, \varepsilon)) \subset B_V$.

(iv) Let $\mathcal{A}$ be an open neighborhood of $X \times I$ in $2^X \times I$. For all $(x,t) \in X \times I$ there exists an open neighborhood $B_{(U_{(x,t)})} \times (t-\varepsilon_t, t+\varepsilon_t)$ of (x_0,t) in $2^X \times I$ such that $B_{(U_{(x,t)})} \times (t-\varepsilon_t, t+\varepsilon_t) \subset \mathcal{A}$. Since I is compact, then for each $x_0 \in X$ there exist $t_1, t_2, \ldots, t_k \in I$ such that $I \subset \bigcup_{j=1}^k (t_j - \varepsilon_{t_j}, t_j + \varepsilon_{t_j})$.

Let $\mathcal{U}_{x_0} = \bigcap_{j=1}^k U_{(x_0,t_j)}$ then $\mathcal{U}_{x_0}$ is open and $x_0 \in \mathcal{U}_{x_0}$. From compactness of X take a finite subcover $\{\mathcal{U}_{x_1}, \mathcal{U}_{x_2}, \ldots, \mathcal{U}_{x_n}\}$. Hence $B_{(\bigcup_{i=1}^n \mathcal{U}_{x_i})}$ is an open neighborhood of X, the canonical copy, in 2^X. Thus there exists $\delta > 0$ such that $X \subset U_\delta(X) \subset B_{(\bigcup_{i=1}^n \mathcal{U}_{x_i})}$. Consequently

$$X \times I \subset U_\delta(X) \times I \subset (B_{(\bigcup_{i=1}^n \mathcal{U}_{x_i})}) \times I \subset \mathcal{A}.$$

Now the proof follows applying the above paragraph to $\mathcal{A} = \widetilde{H}^{-1}(U_\varepsilon(Y))$. □

3. Homotopy and shape properties of U_ϵ

We shall prove some homotopy and shape properties of U_ϵ. First, let us recall some definitions which will be used throughout this section.

Let (X,d) be a compact metric space. $\mathcal{U}$ denotes an open partition of X, a covering formed by disjoint sets, and $\mathcal{P}(X)$ denotes the set of all open partitions of X. Since X is compact we get that any open partition is a finite cover. Recall that a partition $\mathcal{U}_1 \in \mathcal{P}(X)$ is a refinement of another partition $\mathcal{U}_2 \in \mathcal{P}(X)$, and it is denoted by $\mathcal{U}_1 > \mathcal{U}_2$, if for every element $U_i^1 \in \mathcal{U}_1$ there exists an element $U_j^2 \in \mathcal{U}_2$ such that $U_i^1 \subset U_j^2$.

For each $\varepsilon > 0$ we define the set $\mathcal{P}_\varepsilon(X) = \{\mathcal{U} \in \mathcal{P}(X) \mid \{B(x,\varepsilon)\}_{x \in X}$ is a refinement of $\mathcal{U}\}$. Obviously $\mathcal{U} = \{X\} \in \mathcal{P}_\varepsilon(X)$ and since $\{B(x,\varepsilon)\}_{x \in X}$ admits a finite subcover of X, there exists $N_\varepsilon \in \mathbb{N}$ such that N_ε is the minimum number of balls needed to cover X. Hence $\sup\{\mathrm{Card}\{\mathcal{U}\} \mid \mathcal{U} \in \mathcal{P}_\varepsilon(X)\} \leqslant N_\varepsilon$ and so there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$n_0(\varepsilon) = \max_{\mathcal{U} \in \mathcal{P}_\varepsilon(X)} \{\mathrm{Card}(\mathcal{U})\}.$$

Theorem 5. *Let Y be a T_1-space and let X be a compact metric space. If $f : U_\varepsilon \to Y$ is a continuous map then*

$$\mathrm{Card}\big(f(U_\varepsilon)\big) \leqslant \max_{\mathcal{U} \in \mathcal{P}_\varepsilon(X)} \big(\mathrm{Card}(\mathcal{U})\big) = n_0(\varepsilon).$$

Moreover there exists a T_1-space Y and a continuous map $f : U_\varepsilon \to Y$ such that $\mathrm{Card}(f(U_\varepsilon)) = n_0(\varepsilon)$.

Remark 6. Note that if X is connected then $n_0(\varepsilon) = 1$ for every $\varepsilon > 0$ and hence the only continuous maps from U_ε to a T_1-space are the constant maps.

Proof. First of all we are going to show that for every $\varepsilon > 0$ and every continuous map $f : U_\varepsilon \to Y$, with Y a T_1-space, one has that $\mathrm{Card}(f(U_\varepsilon))$ is finite

Suppose that $\varepsilon > 0$ and the map f are fixed. Since $\{B(x,\varepsilon/4)\}_{x \in X}$ is an open cover of X, then there exist $x_1, x_2, \ldots, x_n \in X$ such that $\{B_c(x_i,\varepsilon/4)\}_{i=1,\ldots,n}$ is a finite covering of X, where B_c represents the closed ball. Observe that $B_c(x_i,\varepsilon/4) \in U_\varepsilon$ for all $i = 1,\ldots,n$.

Consider $D \in U_\varepsilon$. For each $y \in D$ there exists $i \in \{1,\ldots,n\}$ such that $y \in B_c(x_i,\varepsilon/4)$. Then we have $\{x_i,y\} \in U_\varepsilon$, $\{x_i,y\} \in \overline{\{x_i\}}$ and $\{x_i,y\} \in \overline{\{y\}}$. Since f is continuous, it follows that $f(\{x_i,y\}) \in f(\overline{\{x_i\}}) \subset \overline{f(\{x_i\})} = f(\{x_i\})$ and $f(\{x_i,y\}) \in f(\overline{\{y\}}) \subset \overline{f(\{y\})} = f(\{y\})$. This implies that $f(\{y\}) = f(\{x_i\})$. Since $D \in \overline{\{y\}}$ it follows that $f(D) = f(\{x_i\})$ since $D \in U_\varepsilon$, $f(D) \in \{f(\{x_1\}),\ldots,f(\{x_n\})\}$. Consequently, $f(U_\varepsilon) \subseteq \{f(\{x_1\}),\ldots,f(\{x_n\})\}$ and thus, $\mathrm{Card}(f(U_\varepsilon)) \leqslant n$.

Now we are going to show that $\mathrm{Card}(f(U_\varepsilon)) \leqslant n_0(\varepsilon)$. Suppose on the contrary that there exists a continuous map $f : U_\varepsilon \to Y$ such that $\mathrm{Card}(f(U_\varepsilon)) = m > n_0(\varepsilon)$. Denote by $\mathrm{Img}(f(U_\varepsilon)) = \{a_1,\ldots,a_m\}$ the image of f. Therefore $\{f^{-1}(a_i) \mid i \in \{1,\ldots,m\}\}$ is an open partition of U_ε and so, since $U_\varepsilon \subset 2_U^X$, we can suppose that

$$f^{-1}(a_i) = \bigcup_{j \in J_i} B_{V_i^j} \subset B_{(\bigcup_{j \in J_i} V_i^j)} \quad \text{for all } i \in \{1,\ldots,m\}.$$

We claim that $\mathcal{V} = \{\bigcup_{j \in J_1} V_1^j, \ldots, \bigcup_{j \in J_m} V_m^j\}$ is an open partition of X. If this is not true, there exist $p,q \in \{1,\ldots,m\}$ and $x \in X$ such that $x \in (\bigcup_{j \in J_q} V_q^j) \cap (\bigcup_{j \in J_p} V_p^j)$. Then there exit $j_1 \in J_q$ and $j_2 \in J_p$ such that $x \in V_q^{j_1}$

and $x \in V_p^{j_2}$, hence $\{x\} \in (\bigcup_{j\in J_q} B_{(V_q^j)}) \cap (\bigcup_{j\in J_p} B_{(V_p^j)})$ which is a contradiction. Moreover we have $\{B(x,\varepsilon)\}_{x\in X}$ is a refinement of $\mathcal{V}$. To see this let $x_0 \in X$ and $x \in B(x_0,\varepsilon)$. Obviously $C = \{x_0, x\} \subset U_\varepsilon$ and, since $C \in \overline{\{x_0\}}$ and $C \in \overline{\{x\}}$, then there exist $p, q \in \{1,\dots,m\}$ such that $f(C) \in f(\overline{\{x_0\}}) \subset \overline{f(\{x_0\})} = \{a_p\}$ and $f(C) \in f(\overline{\{x\}}) \subset \overline{f(\{x\})} = \{a_q\}$. Consequently $p = q$. So $f(x_0) = f(x)$ for all $x \in B(x_0,\varepsilon)$ and it follows that there exists $s \in \{1,\dots,m\}$ such that $B(x_0,\varepsilon) \subset \bigcup_{j\in J_s} V_s^j$. As a result we have that $\mathcal{V} = \{\bigcup_{j\in J_1} V_1^j, \dots, \bigcup_{j\in J_m} V_m^j\} \in \mathcal{P}_\varepsilon(X)$ and that $\mathrm{Card}(\mathcal{V}) = m > n_0(\varepsilon) = \max_{\mathcal{U}\in\mathcal{P}_\varepsilon(\mathbf{X})}(\mathrm{Card}(\mathcal{U}))$ which is not possible. This proves that $\mathrm{Card}(f(U_\varepsilon)) \leqslant n_0(\varepsilon)$.

Now we are going to construct a continuous map $f : U_\varepsilon \to Y$ such that $\mathrm{Card}(f(U_\varepsilon)) = n_0(\varepsilon)$. Let $Y = \{1,\dots,n_0(\varepsilon)\}$. Let $\mathcal{U}_0 \in \mathcal{P}_\varepsilon(X)$ such that $\mathcal{U}_0 = \{\mathcal{U}_0^1, \mathcal{U}_0^2, \dots, \mathcal{U}_0^{n_0(\varepsilon)}\}$. Define the map $f : X \to \{1,\dots,n_0(\varepsilon)\}$ by $f(x) = i$, if $x \in \mathcal{U}_0^i$. For each $C \in U_\varepsilon$, there exists only one $\mathcal{U}_0^p \in \mathcal{U}_0 \in \mathcal{P}_\varepsilon(X)$ such that $C \subset \mathcal{U}_0^p$. Thus $f : X \to \{1,\dots,n_0(\varepsilon)\}$ is extendable over U_ε as

$$\begin{aligned} \hat{f} : U_\varepsilon &\longrightarrow \bigl\{1,2,\dots,n_0(\varepsilon)\bigr\},\\ C &\longrightarrow \hat{f}(C) = i \end{aligned}$$

where $i \in \{1,2,\dots,n_0(\varepsilon)\}$ is such that $C \in \mathcal{U}_0^i$.

As we proved in [2], the map $2^f : 2^X \to 2^{\{1,\dots,n_0(\varepsilon)\}}$ is continuous and obviously $\hat{f} = 2^f|_{U_\varepsilon}$. Consequently $\hat{f}$ is continuous and $\mathrm{Card}(\hat{f}(U_\varepsilon)) = n_0(\varepsilon)$. $\square$

In order to describe the shape of U_ε, we need the following result on shape theory (see [7,8]).

Theorem 7. *Let X, Y be topological spaces and suppose that $f : X \to Y$ is a continuous function. Then the map f induces a shape equivalence if only if for any ANR, P, and any homotopy class $[h] \in [Y,P]$, where $h : Y \to P$ is a map, we have that the assignment*

$$[h] \quad \Longrightarrow \quad [h \circ f]$$

where $[h \circ f] \in [X,P]$, induces a bijection between the corresponding sets of homotopy classes of maps.

Now we can prove

Theorem 8. *Let X be a compact metric space. For every $\varepsilon > 0$, U_ε has the shape of the discrete space $\{1,\dots,n_0(\varepsilon)\}$.*

Proof. Let P be an ANR and suppose that $f_\varepsilon : U_\varepsilon \to \{1,\dots,n_0(\varepsilon)\}$ is an onto continuous function. First of all, let us prove that for every continuous map $g : U_\varepsilon \to P$ we can construct a continuous map $h : \{1,\dots,n_0(\varepsilon)\} \to P$ such that $g = h \circ f_\varepsilon$.

Note that $\mathrm{Card}(g(U_\varepsilon)) = q \leqslant n_0(\varepsilon)$. Suppose that the image of g is $\{a_1,\dots,a_q\}$. Since $f_\varepsilon : U_\varepsilon \to \{1,\dots,n_0(\varepsilon)\}$ and $g : U_\varepsilon \to P$ are continuous functions we have two partitions of U_ε given by

$$U_\varepsilon = \bigcup_{i=1}^{q} V_i \quad \text{with } V_i = g^{-1}(a_i) \text{ for every } i \in \{1,2,\dots,q\},$$

$$U_\varepsilon = \bigcup_{j=1}^{n_0(\varepsilon)} W_j \quad \text{with } W_j = f_\varepsilon^{-1}(j) \text{ for every } j \in \bigl\{1,2,\dots,n_0(\varepsilon)\bigr\}.$$

Let us show now that for every $j \in \{1,\dots,n_0(\varepsilon)\}$ there exists $i \in \{1,\dots,q\}$ such that $W_j \subset V_i$. Suppose, on the contrary, that there exists $j_0 \in \{1,\dots,n_0(\varepsilon)\}$ such that $W_{j_0} \not\subseteq V_i$ for all $i \in \{1,\dots,q\}$. Since $U_\varepsilon = \bigcup_{i=1}^{q} V_i$ and $W_{j_0} \subset U_\varepsilon$ then there exist $i_1,\dots,i_r \in \{1,\dots,q\}$ such that $W_{j_0} \subset \bigcup_{\alpha=1}^{r} V_{i_\alpha}$ and $W_{j_0} \cap V_{i_\alpha} \neq \emptyset$ for all $\alpha \in \{1,\dots,r\}$. Note that $r \geqslant 2$ and $W_{j_0} \cap V_{i_\alpha}$ is an open and closed set of U_ε. Let $(W_{j_0})_\alpha = W_{j_0} \cap V_{i_\alpha}$ for all $\alpha \in \{1,\dots,r\}$ and take

$$U_\varepsilon = W_1 \cup \dots \cup W_{j_0-1} \cup \overbrace{\bigl((W_{j_0})_1 \cup \dots \cup (W_{j_0})_r\bigr)}^{W_{j_0}} \cup W_{j_0+1} \cup \dots \cup W_{n_0(\varepsilon)}.$$

Then we have that U_ε is the union of $(n_0(\varepsilon)-1)+r > n_0(\varepsilon)$ open and closed sets. Now define a function $S: U_\varepsilon \to \{1,\dots,n_0(\varepsilon),\dots,(n_0(\varepsilon)-1)+r\}$ by

$$\begin{cases} S(W_j)=j & \text{if } 1 \leqslant j \leqslant j_0-1,\\ S((W_{j_0})_i)=j_0+(i-1) & \text{if } 1 \leqslant i \leqslant r,\\ S(W_j)=j+(r-1) & \text{if } j_0+1 \leqslant j \leqslant n_0(\varepsilon). \end{cases}$$

It is clear that S is continuous and $\mathrm{Card}(\mathrm{Img}(S)) > n_0(\varepsilon)$. This contradicts Theorem 4. Therefore for every $j_0 \in \{1,\dots,q,\dots,n_0(\varepsilon)\}$ there exists $i_0 \in \{1,\dots,q\}$ such that $W_{j_0} \subset V_{i_0} = g^{-1}(a_{i_0})$.

Now define $h:\{1,\dots,q,\dots,n_0(\varepsilon)\} \to P$ by $h(j)=a_i$ where $W_j \subset g^{-1}(a_i)$. We are going to show that $g = h \circ f_\varepsilon$. To see this, take $C \in U_\varepsilon$ then there exists $j_0 \in \{1,\dots,q,\dots,n_0(\varepsilon)\}$ such that $C \in W_{j_0}$ and so there exists $i_0 \in \{1,\dots,q\}$ such that $W_{j_0} \subset V_{i_0} = g^{-1}(a_{i_0})$. Since $g(C) \in g(V_{i_0}) = g(g^{-1}(a_{i_0})) = a_{i_0}$ and $f_\varepsilon(C)=j_0$ it follows that $h(f_\varepsilon(C)) = h(j_0) = a_{i_0} = g(C)$.

Suppose now that $h \circ f_\varepsilon \simeq h' \circ f_\varepsilon : U_\varepsilon \to P$ ($\simeq$ means homotopic), then there exists a continuous function $\widetilde{H}: U_\varepsilon \times I \to P$ such that $\widetilde{H}(C,0) = h(f_\varepsilon(C))$ and $\widetilde{H}(C,1) = h'(f_\varepsilon(C))$ for every $C \in U_\varepsilon$. Define the function $H:\{1,2,\dots,n_0(\varepsilon)\} \times I \to P$ by $H(i,t) = \widetilde{H}(C_i,t)$ where $C_i \in U_\varepsilon$ with $f_\varepsilon(C_i)=i$.

By Theorem 4 we have that $\mathrm{Card}(\mathrm{Img}(f_\varepsilon))$ and $\mathrm{Card}(\mathrm{Img}(\widetilde{H}_t))$ for all $t \in I$ are finite and by the definition of $n_0(\varepsilon)$ we obtain, in particular, that $\mathrm{Card}(\widetilde{H}_t) \leqslant n_0(\varepsilon)$. Since $\mathcal{R} = \{f_\varepsilon^{-1}(i) \mid i \in \{1,2,\dots,n_0(\varepsilon)\}\}$ is a *maximal* partition of U_ε, we have that $\widetilde{H}_t(C,t) = \widetilde{H}_t(C',t)$ for every $C, C' \in f_\varepsilon^{-1}(i)$, with $i \in \{1,\dots,n_0(\varepsilon)\}$. Therefore H is well defined.

Since H is, obviously, continuous and for each $i \in \{1,\dots,n_0(\varepsilon)\}$ it follows that $H(i,0) = \widetilde{H}(C,0) = h(f_\varepsilon(C)) = h(i)$ and $H(i,1) = \widetilde{H}(C,1) = h'(f_\varepsilon(C)) = h'(i)$ then $h \simeq h'$. So we have proved that the induced map between the sets of homotopy classes of maps is a bijection for every ANR. □

The next result is the first one that points out that the upper semifinite hyperspace of a compact metric space is a good ambient space to detect shape properties. In fact in the next result we will prove that the shape morphism, in the sense of K. Borsuk [5], induced by a continuous function can be detected in this ambient space and using the sets U_ε.

As usual $sh(f) = sh(g)$ means that the maps f and g induce the same shape morphism. When we talk, in the next proposition, about X or Y we refer to their canonical copies inside their corresponding upper semifinite hyperspaces.

Proposition 9. *Let X, Y be compact metric spaces and suppose that $f, g: X \to Y$ are continuous functions. Then $sh(f) = sh(g)$ if and only if f and g are homotopic in $U_\varepsilon = \{C \in 2^Y \mid \mathrm{diam}(C) < \varepsilon\}$ for all $\varepsilon > 0$.*

Proof. Let Q be the Hilbert cube.

Recall that the relation $f \simeq g$ in $B(Y,\varepsilon)$ in Q for all $\varepsilon > 0$, is independent on the topological copy of Q we choose and on the embedding of Y in Q [5]. Hence we are going to choose $Q = \prod_{n=1}^\infty [-1/n, 1/n]$ which is a convex copy of the Hilbert cube in the Hilbert space $l^2 = \{x = (x_n)_{n\in\mathbb{N}} \mid \sum x_n^2 < \infty\}$ with $\|(x_n)\|_2 = (\sum_{n=1}^\infty x_n^2)^{1/2}$. By $\rho(x,y) = \|(x_n - y_n)\|_2$ for all $x, y \in Q$ we mean the induced distance on Q.

Suppose that Y is contained, as a closed set, in the Hilbert cube Q. For each $q \in Q$ we take the set $Y_q = \{y \in Y \mid \rho(y,q) = \rho(q,Y)\}$. In the proof of Proposition 2 we showed that Y_q is closed in Y and that $r: Q \to 2^Y$ define by $r(q) = Y_q$ is a continuous function.

Let $f, g: X \to Y$ be two continuous functions such that $sh(f) = sh(g)$. Then given $\varepsilon/2 > 0$, there exists a continuous map $H: X \times I \to Q$ such that $H(x,t) \in B(Y, \varepsilon/2)$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $(x,t) \in X \times I$ and so $\rho(H(x,t),Y) < \varepsilon/2$. Hence for $y, y' \in Y_{H(x,t)}$ we have $\rho(y,y') \leqslant \rho(y, H(x,t)) + \rho(y', H(x,t)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Consequently, from compactness, $\mathrm{diam}(Y_{H(x,t)}) < \varepsilon$ and then $Y_{H(x,t)} \in U_\varepsilon$ for all $(x,t) \in X \times I$. Now we can define $\widetilde{H}$ by

$$\begin{aligned} &\widetilde{H}: X \times I \longrightarrow U_\varepsilon,\\ &\quad (x,t) \longmapsto \widetilde{H}(x,t) = r\big(H(x,t)\big) = Y_{H(x,t)}. \end{aligned}$$

It is clear that $\widetilde{H}$ is continuous and, since $Y_{f(x)} = f(x)$ and $Y_{g(x)} = g(x)$, it follows that $\widetilde{H}(x,0) = r(H(x,0)) = Y_{f(x)} = f(x)$ and $\widetilde{H}(x,1) = r(H(x,1)) = Y_{g(x)} = g(x)$ for all $x \in X$ then $\widetilde{H}$ is a homotopy between f and g in U_ε.

To prove the other direction we are going to consider again Y as a closed set of Q the same metric convex copy as before. Take in Q the open ball $B_Q(Y,\varepsilon)$. By the properties of the Hilbert cube, there exists a prism K in the sense of K. Borsuk [4] which is a compact ANR neighborhood of Y such that $Y \subset K \subset B_Q(Y,\varepsilon)$. Recall that a prism is a subset of Q which, in particular, is homeomorphic to a product of a finite polyhedron by a Hilbert cube.

Let $f,g : X \to Y \subset K$ be maps. Since K is a compact ANR, then there exists a $\varepsilon' > 0$ such that if $\rho(f(x),g(x)) < \varepsilon'$ then $f \simeq g$ in K [6, p. 111]. Suppose now that f and g are homotopic in U_ε for all $\varepsilon > 0$. Note that we are measuring the diameters by means of the metric ρ.

Let $\varepsilon'' > 0$ be such that $Y \subset B(Y,\varepsilon'') \subset K$. Take $\delta = \min\{\varepsilon',\varepsilon''\}$. Then there exists a homotopy $H : X \times I \to U_\delta$ such that for all $(x,t) \in X \times I$ we have $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Therefore for each $(x,t) \in X \times I$ there exists a neighborhood $V_{(x,t)}$ of (x,t) in $X \times I$ such that $H(V_{(x,t)}) \subset B_{B(H(x,t),\frac{\delta-\operatorname{diam}(H(x,t))}{2})} \subset U_\delta$.

Let $y_1 \in H(\alpha_1,t_1)$ with $(\alpha_1,t_1) \in V_{(x,t)}$ and $y_2 \in H(\alpha_2,t_2)$ with $(\alpha_2,t_2) \in V_{(x,t)}$, then we have $\rho(y_1,y_2) \leqslant \rho(y_1,H(x,t)) + \rho(H(x,t),y_2) + \operatorname{diam}(H(x,t)) < \delta$, then $\operatorname{diam}(H(V_{(x,t)})) < \delta$.

As $\{V_{(x,t)} \mid (x,t) \in X \times I\}$ is an open covering of the compact space $X \times I$ then there exist a finite subcover $\mathcal{V} = \{V_{(x_1,t_1)},\dots,V_{(x_n,t_n)}\}$ such that $\operatorname{diam}(H(V_{(x_i,t_i)})) < \delta$ for $i \in \{1,\dots,n\}$. Consider for each $i \in \{1,\dots,n\}$ the map $\beta_i : X \times I \to \mathbb{R}$ defined by

$$\beta_i(x,t) = \frac{\mathfrak{D}((x,t),((X \times I) - V_{(x_i,t_i)}))}{\sum_{j=1}^n \mathfrak{D}((x,t),((X \times I) - V_{(x_j,t_j)}))}.$$

By $\mathfrak{D}$ we mean a compatible distance in the compact metrizable space $X \times I$. We have that the functions β_i with $i \in \{1,\dots,n\}$, are clearly continuous and $\beta_i(x,t) \neq 0$ for $i \in \{1,\dots,n\}$ if only if $(x,t) \in V_{(x_i,t_i)}$. For each $i \in \{1,\dots,n\}$ fix a point $v_i \in H(V_{(x_i,t_i)})$.

Now we define the function $\widetilde{H}$ by

$$\begin{aligned} \widetilde{H} : X \times I &\longrightarrow Q, \\ (x,t) &\longmapsto \sum_{i=1}^n \beta_i(x,t) v_i. \end{aligned}$$

Let $(x_m,t_m) \in X \times I$ such that $\lim_{m\to\infty}(x_m,t_m) = (x,t)$. Then it follows that

$$\begin{aligned} \rho\big(\widetilde{H}(x,t),\widetilde{H}(x_m,t_m)\big) &= \big\|\widetilde{H}(x,t) - \widetilde{H}(x_m,t_m)\big\|_2 = \left\| \sum_{i=1}^n \big(\beta_i(x,t) - \beta_i(x_m,t_m)\big) v_i \right\|_2 \\ &\leqslant \sum_{i=1}^n \big|\big(\beta_i(x,t) - \beta_i(x_m,t_m)\big)\big| \|v_i\|_2 \leqslant M \sum_{i=1}^n \big|\big(\beta_i(x,t) - \beta_i(x_m,t_m)\big)\big|. \end{aligned}$$

Where $M = \max\{\|v_1\|_2,\dots,\|v_n\|_2\}$.

Since $\lim_{m\to\infty} \beta_i(x_m,t_m) = \beta_i(x,t)$ then $\lim_{m\to\infty} \rho(\widetilde{H}(x,t),\widetilde{H}(x_m,t_m)) = 0$. Therefore $\widetilde{H}$ is continuous.

Let $(x,t) \in X \times I$ be such that it belongs to k of the sets $V_{(x_i,t_i)}$. There is not loss of generality in assuming that $(x,t) \in V_{(x_i,t_i)}$ for $i = 1,\dots,k$. Thus we have $\widetilde{H}(x,t) = \beta_1(x,t)v_1 + \beta_2(x,t)v_2 + \cdots + \beta_k(x,t)v_k$ and $\beta_1(x,t) + \beta_2(x,t) + \cdots + \beta_k(x,t) = 1$.

Take now $z \in H(x,t)$. Then

$$\begin{aligned} \rho\big(\widetilde{H}(x,t),z\big) &= \rho\big(\big(\beta_1(x,t)v_1 + \cdots + \beta_k(x,t)v_k\big),\big(\beta_1(x,t)z + \cdots + \beta_k(x,t)z\big)\big) \\ &= \big\|\big(\beta_1(x,t)(z - v_1) + \cdots + \beta_k(x,t)(z - v_1)\big)\big\| \\ &\leqslant \beta_1(x,t)\|z - v_1\|_2 + \beta_2(x,t)\|z - v_2\|_2 + \cdots + \beta_k(x,t)\|z - v_k\|_2 \\ &< \beta_1(x,t)\delta + \beta_2(x,t)\delta + \cdots + \beta_k(x,t)\delta < \left(\sum_{j=1}^k \beta_j(x,t)\right)\delta < \delta. \end{aligned}$$

Thus $\rho(\widetilde{H}(x,t),H(x,t)) < \delta$ and, since $H(x,t) \subset Y$, we have $\rho(\widetilde{H}(x,t),Y) < \delta$. Consequently, for all $(x,t) \in X \times I$ we obtain $\widetilde{H}(x,t) \subset B(Y,\delta) \subset K \subset B(Y,\varepsilon)$. Moreover as $\rho(\widetilde{H}(x,0),f(x)) < \delta$ and $\rho(\widetilde{H}(x,1),g(x)) < \delta$ then $\widetilde{H}(.,0) \simeq f$ and $\widetilde{H}(.,1) \simeq g$ in K. Consequently $f \simeq g$ in $B(Y,\varepsilon)$. $\square$

The next consequence points out one of the main differences between the homotopy and shape categories for topological spaces. K. Morita [9] proved that every topological space has the shape of some Tychonov space (completely regular + T_1-space). For homotopy, on the contrary, we have:

Theorem 10. *In general U_ε with $\varepsilon > 0$ does not have the same homotopy type as a T_1-space.*

Proof. Let X be a connected ANR such that there exists an essential map $f : X \to X$. (In particular, think of $X = S^1$). Since X, considered as the canonical copy, is dense in 2^X, it follows that U_ε is connected for every $\varepsilon > 0$ and so, by Remark 6, we have that the only continuous maps to a T_1-space are the constants. Therefore suppose that U_ε, for every $\varepsilon > 0$, has the homotopy type of a T_1-space, then U_ε has trivial homotopy type. Consequently for every pair of maps $g, f : X \to X \subset U_\varepsilon$ we have that $f \simeq g$ in U_ε for every $\varepsilon > 0$.

Consequently if f is an essential map and g is a constant map we have $sh(f) = sh(g)$ and this is not possible since X is an ANR. □

For the whole space we have

Proposition 11. *Let X be a non-empty topological space. Then 2^X has the homotopy type of a point.*

Proof. First we define a map $H : 2^X \times I \to 2^X$ by

$$H(C, t) = \begin{cases} X & \text{if } t = 0, \\ C & \text{if } t \in (0, 1]. \end{cases}$$

If we denote by $\mathrm{Id} : 2^X \to 2^X$ the identity map in 2^X and by $F_X : 2^X \to 2^X$ the constant $F_X(C) = X$ for all $C \in 2^X$ then we have $H(C, 0) = F_X(C)$ and $H(C, 1) = \mathrm{Id}_{2^X}(C)$.

We need only to prove continuity of H at $(C, 0)$. Note that $H(C, 0) = X$. Let B_U be a basic neighborhood of X. Then $B_U = 2^X$. Clearly $V = 2^X \times I$ is a neighborhood of $(C, 0)$ such that $H(V) \subset 2^X = B_V$ which proves continuity at $(C, 0)$. □

4. Shape theory

In this section we use the hyperspace with the upper semifinite topology to reinterpret Sanjurjo's description of shape of compact metrizable spaces [10] showing that the hyperspaces, with the upper semifinite topology, are as good ambient spaces as the Hilbert cube used by K. Borsuk in [5].

Let us recall some definitions given by J.M.R. Sanjurjo in [10].

Definition 12. Let X and Y be compact metric spaces. An upper semicontinuous multivalued function $G : X \to Y$ is said to be ε-small if $\mathrm{diam}(G(x)) < \varepsilon$ for any $x \in X$.

Definition 13. Let X and Y be compact metric spaces. Two ε-small upper semicontinuous multivalued functions $F, G : X \to Y$ are said to be ε-multihomotopic, denoted by $F \overset{\varepsilon}{\simeq} G$, if there exists a ε-small upper semicontinuous multivalued function $H : X \times I \to Y$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$ for all $x \in X$.

Definition 14. A multi-net form X to Y is a sequence of upper semicontinuous multivalued functions $\widetilde{F} = \{F_n : X \to Y\}_{n\in\mathbb{N}}$ such that for every $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}$ such that $F_n \overset{\varepsilon}{\simeq} F_{n+1}$ for every $n \geqslant n_0$.

Definition 15. Two multi-net $\widetilde{F}$ and $\widetilde{G}$ are said to be homotopic, $\widetilde{F} \simeq \widetilde{G}$, if for every $\varepsilon > 0$ there is a index $n_0 \in \mathbb{N}$ such that $F_n \overset{\varepsilon}{\simeq} G_n$ for every $n \geqslant n_0$.

J.M.R. Sanjurjo proved in [10] that the category whose objects are the compact metric spaces and the morphisms between them are the homotopy classes of multi-nets (with an special definition of composition) is isomorphic to the shape category of compacta defined by Borsuk in [5].

Now our intention is to reinterpret the category obtained by J.M.R. Sanjurjo using the hyperspace with the upper semifinite topology to obtain that, in fact, what Sanjurjo was doing there was just *to change, as ambient space, the Hilbert cube, used by Borsuk, by the corresponding upper semifinite hyperspace*.

We have the next obvious result.

Proposition 16. *A multivalued function $F : X \to Y$ is an upper semicontinuous multivalued function if and only if the function $f : X \to 2^Y$ defined by $f(x) = F(x)$ is continuous.*

The next definition is similar to the definition of approximative map given by K. Borsuk in [5].

Definition 17. Let X and Y be compact metric spaces. A sequence of continuous functions $\tilde{f} = \{f_k : X \to 2^Y\}_{k\in\mathbb{N}}$ is said to be an approximative map from X to Y if for every neighborhood U of the canonical copy Y in 2^Y there exists $k_0 \in \mathbb{N}$ such that f_k is homotopic to f_{k+1} in U for all $k \geqslant k_0$.

Now we use the base $\mathcal{U} = \{U_\varepsilon = \{C \in 2^X \mid \mathrm{diam}(C) < \varepsilon\}\}_{\varepsilon>0}$ of 2^X to prove the next proposition.

Proposition 18. *A sequence $\widetilde{F} = \{F_n : X \to Y\}_{n\in\mathbb{N}}$ is a multi-net if only if the sequence $\tilde{f} = \{f_n : X \to 2^Y\}_{n\in\mathbb{N}}$ where $f_n(x) = F_n(x)$ for all $n \in \mathbb{N}$ and all $x \in X$ is an approximative map in the above sense.*

Proof. First suppose that $\widetilde{F}$ is a multi-net. Then for every open neighborhood U of Y in 2^Y there exists an $U_\varepsilon \in \mathcal{U}$ such that $Y \subset U_\varepsilon \subset U$. Since $\widetilde{F}$ is a multi-net then for all $n \geqslant n_0$ there exists a ε-small upper semicontinuous multivalued function $H_n : X \times I \to Y$ such that $H_n(x, 0) = F_n(x)$ and $H_n(x, 1) = F_{n+1}(x)$ for all $x \in X$.

Now we can define the function $\widehat{H_n} : X \times I \to 2^Y$ by $\widehat{H_n}(x, t) = H_n(x, t)$ for every $(x, t) \in X \times I$. Then we have that $\widehat{H_n}$ is a continuous function and $\widehat{H_n}(x, t) \subset U_\varepsilon \subset U$ for every $n \geqslant n_0$. Moreover for all $x \in X$ we have $\widehat{H}_n(x, 0) = f_n(x)$ and $\widehat{H}_n(x, 1) = f_{n+1}(x)$. Consequently $\tilde{f} = \{f_n\}_{n\in\mathbb{N}}$ is an approximative map.

Now let us suppose that $\tilde{f}$ is an approximative map. Then for each U_ε there exists a index $n_0 \in \mathbb{N}$ and a continuous function $\widehat{H}_n : X \times I \to 2^Y$ such that for all $n \geqslant n_0$ we have that $\widehat{H}_n(x, t) \in U_\varepsilon$ for all $(x, t) \in X \times I$ and $\widehat{H}_n(x, 0) = f_n(x)$ and $\widehat{H}_n(x, 1) = f_{n+1}(x)$ for every $x \in X$.

So for each $n \geqslant n_0$ we define the multivalued function $H_n : X \times I \to Y$ by $H_n(x, t) = \widehat{H}_n(x, t)$. Since $\mathrm{diam}(H_n(x, t)) < \varepsilon$ for all $(x, t) \in X \times I$ then we have that H_n is ε-small upper semicontinuous multivalued function such that $H_n(x, 0) = F_n(x)$ and $H_n(x, 1) = F_{n+1}(x)$ for all $x \in X$. Consequently $\widetilde{F}$ is a multi-net. □

Following the classical definitions of Borsuk we also have.

Definition 19. Two approximative maps $\tilde{f}$ and $\tilde{g}$, are homotopic, $\tilde{f} \simeq \tilde{g}$, if for each open neighborhood U of the canonical copy Y in 2^Y there exists $n_0 \in \mathbb{N}$ such that f_n is homotopic to g_n in U for every $n \geqslant n_0$.

Using the same arguments as before we have.

Proposition 20. *Let $\widetilde{F}$ and $\widetilde{G}$ be multi-nets and let $\tilde{f}$ and $\tilde{g}$ be the two approximative maps defined by $f_n(x) = F_n(x)$ and $g_n(x) = G_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Then $\widetilde{F}$ and $\widetilde{G}$ are homotopic if only if $\tilde{f}$ and $\tilde{g}$ are homotopic.*

Consequently.

Corollary 1. *The set of all homotopy classes of approximative maps from X to 2^Y is in a bijective correspondence with the set of all homotopy classes of multi-nets from X to Y and then with the set of shape morphisms from X to Y.*

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